Spectral Benchmark for Natural Convection Flow in a Tall Differentially Heated Cavity

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ABSTRACT

In this paper we simulate the flow in the differentially heated cavity with ratio of height to width equal to 8. The method used is spectral and is similar to one used by Xin and Le Quéré in 2002. Instead of using Chebyshev polynomials, we used trigonometric functions to describe spatial distribution of the flow. At Rayleigh number $3.4 \times 10^5$ only one periodic solution was found. Our results differ from the Chebyshev method by a few percent.

1 INTRODUCTION

Natural convection is an important phenomena in nuclear engineering [1]. It represents a passive safety mechanism that is at the core of many reactor designs. Accurate prediction of conditions for a safe operation of a particular reactor component depends on the ability of the used tools to accurately predict natural convection and accompanying phenomena [2]. Benchmark cases such as the case presented in this paper are important for testing these tools.

One of the most basic examples of natural convection flows is a flow in differentially heated rectangular cavity. To better understand the dynamics of flow in such cavity, Christon et al. [3] held a workshop at the First MIT Conference on Computational Fluid and Solid Dynamics in June 2001. During the workshop they compared 31 methods for simulating flows in such cavities. Predominantly the methods used were based on finite volumes and finite elements method. This paper is based on the spectral method used by Xin and Le Quéré [4], that was chosen as the most accurate.

The problem of differentially heated cavity was thoroughly defined by Christon et al. [3]. The sketch of the geometry is shown in figure 1. The cavity is of rectangular shape, where the left wall of the cavity is heated and the right wall is cooled. Upper and lower walls are adiabatic. The cavity has a height to width ratio of 8:1. It is filled with Boussinesq fluid with Prandtl number $Pr = 0.71$. The flow is driven by natural convection.

The shape of the flow in such a setting depends on another non-dimensional property called Rayleigh number $Ra$. Together with above mentioned Prandtl number, ratio of the sides of cavity, boundary conditions and initial state, it fully determines the shape of the flow. Our method uses different initial state as was prescribed by Christon et al. They suggested, that the initial value of
temperature and velocity should be set to zero within the cavity. Our deviation from this suggestion is due to easier implementation of different kind of initial state. Despite this, our method still converges to the same sole oscillatory solution at Rayleigh number $Ra = 3.4 \times 10^5$.

The method used by Xin and Le Quéré is second-order in time. The velocity–pressure coupling is resolved by a projection method. For spatial dimensions they used Chebyshev collocation method. Some parts of their method are explained in the next section, otherwise the details can be found in [4].

The part where this article differs from the mentioned is in the method used to resolve spatial dimensions. Our method is pseudo-spectral, because we use transformations between different spectral representations of functions. This is done to meet the different boundary conditions for temperature and velocities. The method itself is further explained in next section.

2 NUMERICAL METHOD

The non-dimensional equations describing the motion of Boussinesq fluid approximation are[5]

\[
\nabla \cdot \mathbf{u} = 0, \\
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = j \theta - \nabla p + \sqrt{\frac{Pr}{Ra}} \nabla^2 \mathbf{u}, \\
\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \frac{1}{\sqrt{Ra Pr}} \nabla^2 \theta,
\]

where $\mathbf{u} = (u, v)$ represents the non-dimensional velocity, $p$ the non-dimensional deviation from hydrostatic pressure, $\theta$ non-dimensional temperature and $j$ unit vector in the $y$ direction.

The two parameters $Ra$ and $Pr$ are the Rayleigh and Prandtl numbers, respectively, and are
defined as
\[ Ra = \frac{g \beta \Delta T W^3}{\nu \alpha}, \]
\[ Pr = \frac{\nu}{\alpha}, \]
where \( g \) is the gravitational acceleration, \( \beta \) the coefficient of thermal expansion, \( \Delta T \) the temperature difference between the hot and cold walls, \( W \) the width of cavity, \( \nu \) the kinematic viscosity and \( \alpha \) the thermal diffusivity.

Boundary conditions are
\[ u = 0 \quad \text{at} \quad x = 0, 1 \text{ or } y = 0, A, \]  
\[ \theta = \pm \frac{1}{2} \quad \text{at} \quad x = 0, 1 \text{ respectively, and} \quad \frac{\partial \theta}{\partial y} = 0 \quad \text{at} \quad y = 0, A. \]

\( A \) is the height of the cavity (shown in figure 1) in non-dimensional units and is equal to 8. Initial conditions used were \( u = 0 \) and \( \theta = 1/2 - x \).

### 2.1 Time discretization

As noted above, time stepping is done in a similar way as described by Xin and Le Quéré [4]. The derivative with respect to time in equations (1)–(3) is discretized using second order backward differential formula. The convective terms are extrapolated using second order Adams-Bashforth method. Applying this scheme, equations (1)–(3) may be approximated as

\[ \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} + 2 ((u \cdot \nabla)u)^n - ((u \cdot \nabla)u)^{n-1} = \frac{\hat{\beta}}{Ra} \theta^{n+1} - \nabla p^{n+1} + \sqrt{Pr \frac{Ra}{Pr}} \nabla^2 u^{n+1}, \]  
\[ \frac{3\theta^{n+1} - 4\theta^n + \theta^{n-1}}{2\Delta t} + 2 (u \cdot \nabla \theta)^n - (u \cdot \nabla \theta)^{n-1} = \frac{1}{\sqrt{Ra Pr}} \nabla^2 \theta^{n+1}, \]  
\[ \nabla \cdot u^{n+1} = 0. \]

Equation (7) although implicit for \( \theta^{n+1} \) is not coupled with the other two at the current time step and can be solved with variables from previous two steps. The momentum equation (6) and continuity equation (8) are coupled. To solve this coupling the projection method is used. The momentum equation (6) at time step \( n + 1 \) is solved for a different speed \( u^* \) with pressure \( p^n \) from time \( n \)

\[ \frac{3u^* - 4u^n + u^{n-1}}{2\Delta t} + 2 ((u \cdot \nabla)u)^n - ((u \cdot \nabla)u)^{n-1} = \hat{\beta} \theta^{n+1} - \nabla p^n + \sqrt{Pr \frac{Ra}{Ra}} \nabla^2 u^*. \]

Vector field \( u^* \) calculated in this way is not divergence free. To calculate \( u^{n+1} \), \( u^* \) must be corrected by

\[ \frac{3(u^{n+1} - u^*)}{2\Delta t} = -\nabla (p^{n+1} - p^n). \]

To solve this equation, pressure \( p^n \) at time step \( n \) is needed. This can be found in combination with equation (8). By applying divergence to equation (9) one gets

\[ \nabla^2 (p^{n+1} - p^n) = \frac{3\nabla \cdot u^*}{2\Delta t}. \]
that has to be solved for \( p^{n+1} \). By using equation (9), \( u^{n+1} \) may then be retrieved.

The system of equations which are solved at each time step can be rewritten as

\[
\nabla^2 \theta^{n+1} - \lambda \theta^{n+1} = \sqrt{Ra Pr} \left[ \frac{\theta^{n-1} - 4\theta^n}{2\Delta t} + 2(u \cdot \nabla \theta)^n - (u \cdot \nabla \theta)^{n-1} \right], \tag{10}
\]

\[
\nabla^2 u^* - \lambda^* u^* = \sqrt{Ra Pr} \left[ \frac{u^{n-1} - 4u^n}{2\Delta t} + 2((u \cdot \nabla)u)^n - ((u \cdot \nabla)u)^{n-1} - \hat{f} \theta^{n+1} + \nabla p^n \right], \tag{11}
\]

\[
\nabla^2 S = \nabla \cdot u^*,
\]

\[
u^{n+1} = u^* - \nabla S,
\]

\[
p^{n+1} = p^n + \frac{3}{2\Delta t} S,
\]

where \( S \) stands for pressure correction \( 2\Delta t(p^{n+1} - p^n)/3 \) and

\[
\lambda_\theta = \frac{3\sqrt{Ra Pr}}{2\Delta t},
\]

\[
\lambda_* = \frac{3}{2\Delta t} \sqrt{Ra Pr}.
\]

Boundary conditions for temperature \( \theta \) in differential equation (10) are the same as specified with equation (5). For velocity \( u^* \) in equation (11) the boundary condition is the same as for velocity (that is the same as in equation (4)). Boundary conditions for \( S \) in equation (12) are set up, so that velocities \( u^* \) and \( u^{n+1} \) are equal at the boundary.

### 2.2 Spectral method

At each time step equations of the Helmholtz form

\[
\nabla^2 f^{n+1} - \lambda f^{n+1} = g^{n+1}
\]

for field \( f^{n+1} \) have to be solved. The term \( g^{n+1} \) is comprised of all the known quantities form previous time step.

Equations for which the boundary condition is

\[ f = 0 \quad \text{where} \ x = 0, 1 \text{ or } y = 0, A, \]

are solved using sine series along both directions

\[
f^{n+1}(x,y) \approx \sum_{k=1}^{N} \sum_{l=1}^{M} f^{n+1}_{k,l} \sin(\pi k x) \sin(\pi l y/A),
\]

\[
g^{n+1}(x,y) \approx \sum_{k=1}^{N} \sum_{l=1}^{M} g^{n+1}_{k,l} \sin(\pi k x) \sin(\pi l y/A).
\]

The equation (13) is then solved directly and the solutions for coefficients \( f^{n+1}_{k,l} \) for \( k = 1, 2, \ldots, N \) and \( l = 1, 2, \ldots, M \) are

\[
f^{n+1}_{k,l} = -\frac{g^{n+1}_{k,l}}{\pi^2 k^2 + \pi^2 l^2 / A^2 + \lambda}.
\]

(14)
The Poisson equation (12) for pressure correction $S$ is treated similarly. The function is approximated as a cosine series over both spatial directions to adhere to boundary conditions. The direct solution is then in the same form as equation (14) with $\lambda$ set to zero. It also includes coefficients $f_{n+1}^{0,k}$ for $k = 1, 2, \ldots, N$ and $f_{n+1}^{l,0}$ for $l = 1, 2, \ldots, M$ calculated by equation (14) with appropriately inserting zeroes. The coefficient $f_{n+1}^{0,0}$ is set identically to zero.

The temperature equation (10) has a different boundary condition. The solution is found as

$$\theta^{n+1}(x, y) \approx \theta^B(x, y) + \sum_{k=1}^{N} \sum_{l=0}^{M} \theta_{k,l}^{n+1} \sin(\pi k x) \cos(\pi l y/A),$$

where $\theta^B(x, y) = 1/2 - x$ is a function that ensures the function $\theta^{n+1}(x, y)$ adheres to boundary conditions. Direct solution is then of the form

$$\theta_{k,l}^{n+1} = -g_{k,l}^{n+1} + \lambda \theta B_{k,l}/(\pi^2 k^2 + \pi^2 l^2/A^2 + \lambda \theta),$$

for $k = 1, 2, \ldots, N$ and $l = 0, 1, \ldots, M$. $B_{k,l}$ are the coefficients in sine–cosine series of function $\theta^B(x, y)$ that may, for same $l$s and $k$s, be written as

$$B_{k,l} = A_{2k\pi}(1 + (-1)^k)\delta_{l,0},$$

where $\delta_{l,0}$ is the Kronecker delta symbol. Expansion of constant term in the function $\theta^B$ into sine series is probably the greatest source of error and the reason for a need to use many orders to achieve satisfactory results. In cases where parameter $\lambda$ is large, the solution for every second coefficient $\theta_{k,l}^{n+1}$ drops to zero as $\sim 1/k$.

### 2.3 Calculating the right hand sides

Calculating the right hand sides of equations involves calculation of derivatives. Each derivative is calculated with help of series expansion, but a sine series expansion after derivation becomes a cosine series expansion. The added zeroth order term in cosine series expansion has to be zero.

Products of functions on the right hand sides are calculated in physical space. This makes the method pseudo-spectral. Each function involved in product on the right hand of equations (10) and (11) is transformed to physical space using a combination of fast sine or cosine transform. In physical space the functions are multiplied and then the product is transformed back to spectral space using the appropriate transform. No de-aliasing was performed.

Functions on the right hand sides that are not in appropriate series expansion for addition and for calculating a direct solution by equations (14) or (15) are transformed into physical space and then transformed back into appropriate series expansion (sine series is transformed to cosine series or reverse).

### 3 RESULTS

We performed simulations of natural convection in a cavity with a method used by Xin and Le Quéré which they describe in [4] as well as with a method which we devised ourselves. As noted in the Introduction as well as in the derivation of our method, we deviated from suggested initial state in the implementation of our method. For temperature, we used the falling linear gradient from...
hot to cold wall as our initial state (that is, all the spectral coefficients are zero, but the boundary condition is implemented inherently into the method, so the initial state is still non-zero). However, for implementation of the method which is described by Xin and Le Quéré, we used initial state where all the variables are zero (as is suggested by Christon et al. and described by Xin and Le Quéré). Like other participants of the workshop, we too observed only one oscillatory solution at Rayleigh number $Ra = 3.4 \times 10^5$ with both methods. The solution is symmetric. This is confirmed through use of the parameter $\epsilon_{12}$ which is defined as a sum of temperatures at monitor points designated as point 1 and point 2 by Christon et al. (see figure 1)

$$\epsilon_{12} = \theta_1 + \theta_2,$$

that oscillated around zero with the amplitude of the order of $10^{-16}$.

![Figure 2: Ten periods of oscillations of temperature $\theta$ at point 1 and Rayleigh number $Ra = 3.4 \times 10^5$ are shown with a solid red line. Blue solid line represents the average temperature as calculated by Xin and Le Quéré. Gray dotted lines designate their peak to valley amplitude centred around the average (the blue line). Blue dashed lines designate the same amplitude, but instead moved slightly (order of $10^{-3}$) to account for the fact that oscillations are not linear. For our method $N = M = 252$ orders were used.]

After the initial transient the temperature of the solution at point 1 oscillates around the average temperature $\bar{\theta}_1$ with the amplitude $\theta'_1$. This is shown on figure 2. The particular solution is calculated for Rayleigh number $Ra = 3.4 \times 10^5$ and Prandtl number $Pr = 0.71$ with $N = M = 252$ modes in each direction. The time step used was $10^{-3}$.

Figure 3 shows the convergence of the average temperature and the amplitude of temperature oscillations with regard to the modes in $y$ direction for the same Rayleigh and Prandtl number as above. The dashed line on both figures represent the values obtained by Xin and Le Quéré [4, Table II, smallest time step]. The values did not change significantly with the change of time step size or the change of modes in $x$ direction. It can be observed, that the solution using our method has different average and amplitude of oscillations. Because of asymptotic behaviour of spectral coefficients mentioned in previous section, the number of orders in $x$ direction must be higher than
Figure 3: Variation of the average $\theta_1$ (left) and the amplitude $\theta'_1$ (right) of temperature by number of modes in $y$ direction. The number of modes in $x$ direction is held constant at $N = 252$ modes. The dashed lines show reference values.

with the approach of Xin and Le Quéré. They use at most 48 Chebyshev polynomials in the $x$ direction and at most 180 Chebyshev polynomials in $y$ direction.

Table 1 holds some of the parameters required by Christon et al. at the workshop. Shown are the averages and amplitudes for temperature, velocities at monitoring point 1 and the period of temperature oscillations $\tau_\theta$, which is the same for other parameters. The average Nusselt numbers and amplitudes at both walls are given with average and amplitude of differences of pressure among monitor points ($\Delta p_{ij} = p_i - p_j$). Values were obtained for a simulation with the same values of parameters that were used for figure 2. The analysis of parameters was performed for a time duration of about 500 units or 146 periods.

In addition to above results we also performed simulation with original method used by Xin and Le Quéré. The results are presented in [6] and [7]. Our results using their approach differed relatively from theirs by an order of $10^{-4}$ while using the same mesh sizes ($48 \times 180$) and time steps ($\sim 8 \times 10^{-4}$). With their approach we also tried to map the different solutions at particular Rayleigh numbers [7] based on the period of the solution.

4 CONCLUSION

We simulated the flow in a tall differentially heated cavity filled with Boussinesq fluid. We used pseudo-spectral method based on the method of Xin and Le Quéré [4]. Similar to the participants of the First MIT Conference on Computational Fluid and Solid Dynamics in June 2001 [3], we obtained only one periodic solution at Rayleigh number $Ra = 3.4 \times 10^5$. The results of Xin and Le Quéré were chosen to be the most accurate by the organisers of the workshop. Our results differ from these reference results by a few percent. The number of modes necessary to achieve this accuracy is rather high, due to unfortunate choice of basis functions for these particular boundary conditions. Nevertheless, these results can already be used to test the accuracy of some codes for simulating fluid flows.

REFERENCES

Table 1: Local and global results for simulation of natural convection in a tall cavity at Rayleigh number $3.4 \times 10^5$. $N$ and $M$ where taken to be equal to 252. Time step of $10^{-3}$ was used. Results are compared to that of Xin and Le Quéré [4, Table II, minimal time step].

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<th>$u'_1$</th>
<th>$\bar{v}'_1$</th>
<th>$v'_1$</th>
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