ABSTRACT

The first step towards finding an approximate sheath solution in the limit $\varepsilon = \lambda D/L \to 0$ is to find the value of the alpha coefficient with the corresponding coefficient in the form of the sheath edge singularity $(\Phi_s - \Phi) \to C(x_s - x)\alpha$ describing the limiting potential variation in front of the sheath edge $(x_s, \Phi_s)$. Since the analytic solution is not difficult to obtain for the cold ion-source case, we can safely assume that alpha in this case is exactly $1/2$. For the finite ion-source Riemann [J.Phys.D: Appl.Phys. 24, 493 (1991)] expects $\alpha = 2/3$, but this has never been confirmed.

Approximations on the high grading mesh near the sheath edge applied to a large number of normalized ion-source temperatures ($0.01 \leq T_n \leq 100$) show that Riemann’s expectation is correct except in a very narrow region ($T_n < 0.1$). Additionally, we show behavior of $C$ ‘constant’, which has failed to attract attention so far. In the case of warm ion source the $C$ coefficient is in fact a function of the ion source temperature. In the region of high temperatures this coefficient remains constant. Our result will help proceed with “mission impossible” of finding a proper plasma, sheath and intermediate region analytic expression.

1 INTRODUCTION

The problem of the potential profile shape near the plasma boundary is an important one in the plasma theory and its application. It has been shown by Riemann [2] that in kinetic approach this problem can be solved successfully only for cold ion sources but that in the case of finite ion-source temperatures the problem is extremely stiff because of the mathematical structure of the basic integral-differential equation which does not permit any reasonable approximative solution.

Complete numerical plasma and sheath equation was first obtained by Self [3] many years ago, for the case of the zero initial ion-source temperature, while the case of finite ion-source temperatures was tackled by Robertson [4] in 2009, yet only on the equidistant computational
grid with rather low resolution. The solution with a considerably increased number of computational cells enabling extremely high resolution while approaching the wall was obtained last year by Kos and Jelić et al. [5, 6]. Therefore the problem seems to be completely resolved. In spite of this, however, approximate analytic solutions for plasma region and sheath region still remain of high interest to plasma physics, not only from the theoretical but also from the pragmatic point of view because this approach helps us to define a reasonable plasma-sheath transition via constructing the analytic solution which smoothly patches plasma and sheath branches of curves. Confirming the validity of Riemann’s scaling rules (also known as “similarity rules”) is a prerequisite for constructing the asymptotic analytic solution for non-zero (high enough) ion source temperatures. In K.-U. Riemann’s paper [1] the results are obtained by means of a reasonable but not strictly proven assumption (see the discussion following Eq. (75) in [1]). Ref. [1] also expresses ‘expectation on the asymptotic behavior’ of the potential [see the discussions in the paragraph just above Eq. (103) and Section 5 and discussions just above Eq. (103)]. According to Riemann’s explanation presented recently [2] ‘The structure of the plasma-sheath transition for models with hot ion source was never analyzed!’. 'The analysis is extremely involved due to the following difficulties:

- The self-consistent electric field is described by an integro-differential equation of the Fredholm type.
- The problem is coupled with an eigenvalue problem caused by the plasma balance.
- The plasma approximation is impeded by the sheath edge singularity.'

As an answer to recognized difficulties our code [5] provides the analytic-numerical solution of an integral equation with a special kernel emerging from the physical scenario modeled first by Bissell and Johnson in 1987 [7]. The essential parameter of their problem is the ion source temperature emerging from the Maxwellian shaped ion-source velocity distribution (VDF) at the place of their creation i.e., “birth”. In contrast to limitations of previous models [7, 8] we are able to provide high resolution at the sheath edge from which the form of singularity is questioned.

2 BACKGROUND

The geometry of the symmetric T&L model in one-dimensional (plane) geometry with potential $\Phi(x)$ is schematically shown in Fig. 1. The plasma center at $x = 0$, walls at $x = \pm L$. $\Phi_s$ is the potential of the sheath edge, $\Phi_w$ is the wall potential. The electrostatic potential $\Phi(x)$ is assumed to be monotonically decreasing (for $x > 0$) and is defined to be zero at $x = 0$. Here we just point out that the central quantity of interest in the present work is the potential profile in so-called $\varepsilon = 0$ limiting case, where $\varepsilon$ is the ratio of the electron Debye length to appropriate characteristic length of plasma. For the Maxwellian ion-source velocity distribution the system of equations in a normalized form turns to the Fredholm-type integral equation

$$\frac{1}{B} = \int_0^1 \exp \left[ \left( 1 + \frac{1}{2T_n} \right) \left( \Phi - \Phi' \right) \right] K_0 \left( \left| \frac{\Phi - \Phi'}{2T_n} \right| \right) dx', \quad (1)$$

Figure 1: The geometry and coordinate system.
where kernel $K_0$ is a modified zero-Bessel function. For the normalized system length $L = 1$ the sheath edge is located at the reference point $(x_s = 1, \Phi_s)$. The role of this reference point is usually realized as the sheath edge [1, 9, 10]. The term “sheath edge” refers exclusively to the asymptotic case $\varepsilon \to 0$. In this asymptotic case the sheath edge is mathematically defined by a singularity of the (quasi-neutral) solution. Floating wall potential $\Phi_w$ is contained within ‘eigenvalue’ $B$ related to the ionization length [6] and can be calculated for particular gas properties $m_e/m_i$ with

$$\exp(\Phi_w) = 2\pi \sqrt{\frac{m_e}{m_i}} \sqrt{T_n} B \int_0^1 dx' \exp[\Phi(x')] .$$

(2)

3 NUMERICS

The main question concerning numerical determination of sheath edge singularity is focused on the quality of potential profiles $\Phi(x)$. Can we find safely the power alpha in the formula

$$(\Phi_s - \Phi) \to C(x_s - x)^\alpha$$

(3)

describing the limiting potential variation in front of the sheath edge $(x_s, \Phi_s)$ for $T_n > 0$ (that means $T_n = O(1)$). We expect $\alpha = 2/3$ independently on the detailed value of $T_n$. Secondly, what algorithm is appropriate for finding the sheath edge singularity. Fig. 2 shows a detail of central interest with axes as defined by Eq. (3). Origin $(0,0)$ is the sheath edge $(x_s, \Phi_s)$. Two distinct models are presented. The analytical solution for ‘cold’ $T_n = 0$ T&L model and finite ion-source temperature model with $T_n = 1$. We can safely assume (because we know the the analytic solution) that alpha is exactly $1/2$ for $T_n = 0$. The analytical potential profile is discretized to correspond to the discretization used in our program code for finite ion-source temperatures [5, 11]. High resolution grading near the sheath edge is required for precise treatment in the area of interest. Fig. 2 shows such increasing density with detail width $w = 0.0001$ with more that 50 discretization points in the selected range.

Recently we have upgraded our code with the piecewise Lagrangian interpolation of order 2 or 3 in the areas of mild $\Phi(x)$ gradients, so we could perform iterations with the same or better accuracy. Upgrade also enabled wider ion-source temperature ranges, especially in the limit

Figure 2: Sheath edge detail for ‘cold’ $T_n = 0$ and ‘warm’ $T_n = 1$ ion-source model.
$T_n \to 0$ where we previously experienced instabilities due to prolonged integrations intervals caused by $1/T_n$ singularity [see Eq.(1)] in the kernel. Estimation of $\alpha$ was performed by a non-linear model fitting with a different number of discretization end-points. As expected the width of approximation near the sheath edge should be sufficiently small to characterize singularity and sufficiently large to minimize uncertainty. To analyze this properties both cold and warm ion-source models are matched under the same ‘numerical’ conditions.

### 3.1 Cold ion-source model

For zero ion-source temperature $T_n = 0$ exact solution [12, 10] describing a collision-free planar discharge model containing imaginary error function [13] is expressed as

$$x(\Phi) = \frac{\sqrt{2}}{\pi} \left( \sqrt{-\Phi} \exp(-\Phi) + (1 + 2\Phi) \int_0^{\sqrt{-\Phi}} \exp(t^2) dt \right).$$

(4)

A valid $\Phi$ range is from 0 to $\Phi_s = -0.8540326565981972$, which gives the system length of $L_0 = x(\Phi_s) = 0.572136376739$. For comparison with normalized system length $L = 1$, inverse function $\Phi(z)$ can be numerically solved by finding the root of $z - x(\Phi)/L_0 = 0$. Although Eq. (4) can be evaluated to arbitrary precision, we took approach that is compatible with our code. Namely, using higher precision for Eq. (4) and saving a potential profile in the file with compatible precision. To simulate high grading near the sheath edge used in the warm case the potential curve is positioned at the following discrete positions

$$x_i = \left( 1 - \left( 1 - \frac{i}{np - 1} \right)^{\lambda_2} \right)^{\lambda_1}, \quad i = 0, 1, \ldots, np - 1,$$

(5)

where number of points $np$ and grading at endpoints $\lambda_1$ and $\lambda_2$ should be similar to those used in warm case $T_n > 0$.

![Figure 3](image-url)  

Figure 3: Approximation of power $\alpha$ for $T_n = 0$ with grid $np = 2401$ points and density $\lambda_1 = 1$, $\lambda_2 = 2.4$. The width of approximation shown in (a) with the same number of points used in approximation $n$ as in grid scale (b). The inset graphs show detail behavior for evaluation of the $\alpha$ estimation criterion. The dashed line in (b) suggests the extrapolation criterion.

Figure 3 shows behavior of our approximation algorithm applied to the ‘cold’ case, for which the ‘exact’ potential profile can be evaluated at arbitrary precision. When using long
double machine precision, approximation errors are inevitable. At least a 30 grid point must be used for correct $\alpha$ estimation with a given number of grid points and density. Fig. 3(a) and inset detail in terms of approximation width $w = x_s - x$. Fig. 3(b) shows the same details in terms of number of approximation points $n$ for selected grading. The grid scale in Fig. 3(b) shows that at least 30 grid points must be used to estimate $\alpha = 1/2$. This corresponds to the approximation width of $w = 0.00003$ as shown in the inset graph of Fig. 3(a). The inset graphs also show that, when ruling out tiny range where uncertainty is high, evaluation close to the theoretical value is possible for a large number of grid points. This is especially true for grid scale (b) where also extrapolation criterion can be applied.

3.2 Warm case

While VDF for the cold ion-source is the Dirac $\delta$-function, for the finite ion-source temperatures $T_n > 0$ a variety of VDFs are possible. For the $\alpha$-approximation we applied the Maxwellian ion-source, as results were readily available with various grid setups, so we could also test the grid invariance. It turned out that most of the potential profiles near the sheath edge were not accurate enough for $\alpha$ to be reasonably estimated. While all previous parameters $(\Phi_s, B)$ converged within $10^5$ iterations, the sheath edge detail, from which $\alpha$ is approximated, needs an order of magnitude additional iterations. Fig. 4 shows such $\alpha$ convergence for $T_n = 1$,

![Figure 4: Convergence of $\alpha$ and a corresponding optimal number of approximation points $n$.](image_url)

which took more than three months of computation time on 16 processors compute node and was stopped when convergence to $2/3$ was observed and sufficiently precise results for $\alpha$ were obtained. The calculation of the potential profiles for the whole temperature range and different grids took more than 700000 processor hours. Fig. 5 shows a similar decreasing function as for the ‘cold’ case in Fig. 3. In contrast with the cold case, we observed here a higher gradient and systematic deflection that underestimates $\alpha$ in the ‘uncertainty’ range. The dash-dot line in Fig. 4 shows that the number of approximation points decreases and approaches that of the cold case. The minimal approximation width is thus dependent on the ‘quality’ of the potential profile. We took advantage of the convex function near the sheath edge and resolved with the simple criterion $\alpha = \alpha_{\text{max}}$, which selects the deflection point. When ruling out the ‘minimal’ width, any other estimation criterion can be used to determine the sheath edge singularity in the limit. As seen from the inset graphs, the approximation width is still quite large.
Figure 5: Approximation of power $\alpha$ for $T_n = 1$ with grid $n_p = 2401$ points and density $\lambda_1 = 1, \lambda_2 = 2.4$. The width of approximation shown in (a) with the same number of points used in approximation $n$ as in grid scale (b). The inset graphs show in detail the behavior for the evaluation of the $\alpha$ estimation criterion.

We can conclude that in the limit the power law holds and that problems with the appropriate width when approaching $w \to 0$ are related to numerical uncertainty. Thus, we can find safely the power alpha with proposed approximation and perform even better if taking into account $\alpha(w)$ variation.

4 RESULTS

All numerical results presented are for the Maxwellian ion-source VDF. Analytical expectations as to the sheath edge singularity are ion-source independent. Preliminary results with the ‘water-bag’ VDF confirm such expectations. Dependence of $\alpha$ on $T_n$ is shown in Fig. 6.

Figure 6: Dependence of power $\alpha$ on ion-source temperature $T_n$ in logarithmic scale.

The logarithmic scale for $T_n$ is used as otherwise transition from $1/2$ to $2/3$ would be sharp with a large $T_n$ width. We used different grids to prove invariance of the $\alpha$ on the grid setup. For $T_n \geq 1$ we see that approximation came within 1% to the theoretical value of $\alpha = 2/3$. The main question remains in the transitional area $T_n \leq 0.1$ where we observed a gradual drop of $\alpha$.
to the theoretical limit for $\alpha_0 = 1/2$. It should be noted, that a potential profile for $T_n = 0$ is simulated from the analytical T&L solution and not obtained from our code. This additionally supports the reliance of the approximation approach. The quality of the potential profiles in this area still needs to be improved by forcing additional iterations that will stabilize $\alpha$. Precise behavior in this range also requires additional points. Transition range $T_n < 0.1$ coincides with the simulations [14] using cool Maxwellian ion source. Figure 7 shows the behavior of ‘constant’

![Figure 7: Dependence of constant C on ion-source temperature T_n in the logarithmic scale.](image)

$C$, which in the case of warm ion source is in fact a function of the ion source temperature. In the region of high temperatures this coefficient can be considered as a constant.

5 CONCLUSION

Our numerical procedure showed a good agreement with Riemann’s $\alpha = 2/3$ estimation for $T_n > 0$. The answer to the question where and how $\alpha = 1/2$ for ‘cold’ ion-source transforms into ‘warm’ $\alpha = 2/3$ sheath-edge singularity is given in Fig. 6. We can safely conclude that $\alpha = 2/3$ is valid for $T_n > 0.1$ ion-source temperatures. Precise $\alpha$ behavior in the narrow region $0 < T_n < 0.1$ is yet to be determined by further improvements in our code and gaining additional computational resources. Numerical approximations showed that $\alpha$ evaluation from potential profiles can serve as a powerful criterion to determine the quality of the profile near the sheath edge, where most of the research interest is focused. How this relates to the derived results like VDF and higher moments will be determined in our future work.

This work gives a foundation for the construction of the intermediate region that will fill the gap between plasma and sheath solutions for ‘finite’ ion-source temperatures. Without gaining this experience on the sheath edge approximation we could not have encountered deficiencies in our previous results, which could lead to misinterpreted derived works in describing the intermediate region. The only question that we are unable to answer safely at the moment is the question on the validity of the power law $C(x_s - x)^\alpha$ itself. For that we would need the inverse numerical model $x(\Phi)$ [instead of $\Phi(x)$], with high precision that could go beyond the physical limits $x > x_s$. This theoretical answer on parabolic sheath-edge singularity is known for $T_n = 0$, while for $T_n > 0$ it remains open.
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